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Limits of Continuum Aerodynamics

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At very low densities or in the flow about very small objects, air cannot be modeled as a fluid continuum. In such cases, classical continuum theory does not apply. In typical aerodynamics textbooks, the limitations of continuum theory are discussed only briefly or not at all. Here, an improved rule for approximating the limits of continuum theory is presented. The effects of Knudsen number, Reynolds number, and Mach number are considered. The criterion is based on a 1% deviation between results predicted from continuum theory and results predicted from the kinetic theory of gases.

I. Introduction

THE analytical methods used in classical aerodynamics treat the atmosphere as a fluid continuum. According to the classical scientific definition, a fluid is a substance that deforms continuously under the action of a shear stress. However, what is commonly called *still air* is comprised of very small moving molecules separated by relatively vast amounts of empty space. The air is said to be still only because the statistical average velocity of the molecules is zero. When a particular molecule collides with a moving object, its velocity is changed. As many molecules collide with the moving object, the statistical average velocity of the molecules changes. This change in average velocity is manifested as a deformation of the air. However, strictly speaking, the deformation of the air is not continuous but comes as a series of discrete impulses. Nevertheless, in most commonly encountered situations such deformation can be approximated accurately as being continuous, because the number of collisions with the object per unit time is very large. Thus, air at the altitudes normally encountered in flight can usually be modeled accurately as a fluid continuum.

At very high altitudes and/or when considering flow about very small objects, air cannot be modeled as a continuum, and in such cases, the mathematical formulations used in classical aerodynamics do not apply. A dimensionless parameter that is typically used to define the boundary of continuum flow is the Knudsen number Kn, which is defined to be the ratio of the molecular mean free path λ to the smallest significant dimension of the flow geometry d (i.e., $Kn \equiv \lambda/d$). The molecular mean free path is the mean distance that a molecule travels between collisions with neighboring molecules. For an object moving through the atmosphere, the accuracy of continuum theory also depends on the Mach number.

Strictly speaking, continuum theory applies only in the limit as the Knudsen number approaches zero. However, the molecular mean free path for air at standard sea level is approximately 2.1×10^{-7} ft. At higher altitudes the mean free path increases, and at 50,000 ft the mean free path is about 1.2×10^{-6} ft. Even at an altitude of 200,000 ft, the mean free path is still less than 9.1×10^{-4} ft. Thus, except for very high altitudes, and/or for flow about very small objects, atmospheric air can be approximated as a fluid continuum.

Nevertheless, an engineer should never lose sight of the fact that the continuum theory of aerodynamics is only a mathematical model used to approximate the behavior of a collection of molecules. There are important applications, such as spacecraft launch and reentry and flow about small aerosol particles, which cannot be analyzed accurately using continuum theory. At an altitude of 400,000 ft, the molecular mean free path for atmospheric air is nearly 24 ft. Thus, at this altitude, atmospheric air cannot be modeled as a fluid continuum. During spacecraft launch and reentry, vehicles like the space shuttle must pass through a region of the atmosphere where continuum theory does not apply and yet the forces exerted on the vehicle by the surrounding air cannot be ignored.

At low altitudes, airflow about an object having a minimum significant characteristic dimension of 1 m would result in a Knudsen number on the order of 10^{-7}. As the altitude increases or the size of the object decreases, the Knudsen number becomes larger. When the Knudsen number becomes large enough, continuum theory first begins to break down with failure of the no-slip boundary condition at the interface between the air and the object. The value of the Knudsen number at which the no-slip boundary condition first begins to fail depends on the airspeed. For objects as large as manned aircraft, continuum theory first begins to break down with failure of the no-slip boundary condition near the leading edge of objects moving through the atmosphere at high altitudes and hypersonic airspeeds. For aerosol particles less than about 10 to 20 μm in diameter, the no-slip boundary condition may not be applicable even for low airspeeds at sea level.

In most commonly used undergraduate engineering textbooks on aerodynamics, the limitations of continuum theory are discussed only briefly or not at all (see, for example, Anderson [1], Bertin and Cummings [2], Kuethe and Chow [3], and McCormick [4]). Furthermore, there appears to be some disagreement in the undergraduate engineering literature regarding where the practical boundaries of continuum theory actually fall. For example, Anderson [1] states, "If λ is orders of magnitude smaller than the scale of the body measured by d, then the flow appears to the body as a continuous substance." In contrast, Bertin and Cummings [2] state, "Although there is no definitive criterion, the continuum flow model starts to break down when the Knudsen number is roughly of the order of 0.1." An even more liberal opinion is rendered by Fox and McDonald [5], who state, "The concept of a continuum is the basis of classical fluid mechanics. . .However, it breaks down whenever the mean free path of the molecules becomes the same order of magnitude as the smallest significant characteristic dimension of the problem."

It is important that engineers keep in mind the limitations of the mathematical formulations that are used to solve engineering problems. An improved rule for approximating the limits of continuum theory would be helpful in this regard. In the present paper the effects of Knudsen number, Reynolds number, and Mach number on the limits of continuum theory are examined and discussed at a level that would be useful for both engineering students and practicing engineers.

Because continuum theory strictly applies only in the limit as the Knudsen number approaches zero, when the Knudsen number exceeds some limiting value, we must account for the fact that a gas is made up of individual molecules in a state of constant motion. In such

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cases, the macroscopic motion of the gas and the forces exerted on any bounding solid surface must be analyzed from a statistical consideration of the motion of the molecules. The related field of science that deals with the statistics of molecular motion in gases is called the *kinetic theory of gases* or, less commonly, *rarefied gas dynamics*. The kinetic theory of gases is a well-established scientific discipline, which dates back to Maxwell [6–8]. The reader who may not be familiar with kinetic theory could begin with the classic work by Jeans [9]. Other useful textbooks on the topic include Brush [10], Liboff [11], Present [12], Struchtrup [13], and Vincenti and Kruger [14].

II. Creeping-Flow Limit

From continuum theory applied to the flow about an object moving through the atmosphere, the traditional aerodynamic coefficients for lift and drag are typically found to be functions of flow geometry, Reynolds number, and Mach number. For very-low-density flows where continuum theory does not apply, the kinetic theory of gases predicts that these same aerodynamic coefficients are functions of flow geometry, Knudsen number, and Mach number. Because continuum theory is the limiting case for kinetic theory as the Knudsen number approaches zero, this requires the existence of a fundamental relation between the Reynolds number, Mach number, and Knudsen number. Kinetic theory provides the required result in the form of an analytical relation between the dynamic viscosity and the molecular mean free path for an ideal gas [9–14]:

$$\mu = \sqrt{2RT/\pi}\rho\lambda \tag{1}$$

where R is the ideal gas constant, T is absolute temperature, and ρ is the gas density. The speed of sound for an ideal gas is given by

$$a = \sqrt{\gamma RT} \tag{2}$$

where γ is the traditional specific-heat ratio, which is 1.4 for atmospheric air. Combining Eqs. (1) and (2), the kinematic viscosity can be expressed in terms of the molecular mean free path and the speed of sound:

$$\nu \equiv \mu/\rho = \sqrt{2/(\pi\gamma)}\sqrt{\gamma RT}\lambda = \sqrt{2/(\pi\gamma)}a\lambda \tag{3}$$

In view of Eq. (3), the Reynolds number Re can be expressed as

$$Re \equiv \frac{Vd}{\nu} = \sqrt{\pi\gamma/2}\frac{Vd}{a\lambda} = \sqrt{\pi\gamma/2}\frac{V/a}{\lambda/d} = \sqrt{\pi\gamma/2}\frac{M}{Kn}$$

where V is the airspeed and M is the Mach number (i.e., $M \equiv V/a$). Thus, the Knudsen number for an ideal gas can be computed directly from the Mach number and Reynolds number:

$$Kn = \sqrt{\pi\gamma/2}M/Re \tag{4}$$

For the limit as the Reynolds number approaches zero, accurate analytical solutions are available from both continuum theory and kinetic theory. Such solutions are commonly referred to as creeping-flow solutions. For example, a solution for the drag on a sphere in creeping flow as predicted from continuum theory was first published by the famous British mathematician and physicist George Gabriel Stokes in 1851 [15]. This remarkably simple result can be written as

$$D = 3\pi\mu Vd \tag{5}$$

where D is the drag on the sphere and d is the sphere diameter. Results obtained from the kinetic theory of gases are compared with this continuum solution in Fig. 1. The analytical results presented in Fig. 1 are taken from Phillips [16] and the experimental results are from Millikan [17–19]. The six-moment solution shown in Fig. 1 agrees very well with experimental data for aerosol particles of various shapes, provided that a volume-weighted mean diameter is used for d. Note that significant departure from continuum theory begins in the Knudsen-number range 0.001–0.01 and produces a 20% reduction in drag at a Knudsen number of approximately 0.1. At a Knudsen number of 1.0, the drag is reduced by more than 75% relative to that predicted from continuum theory.

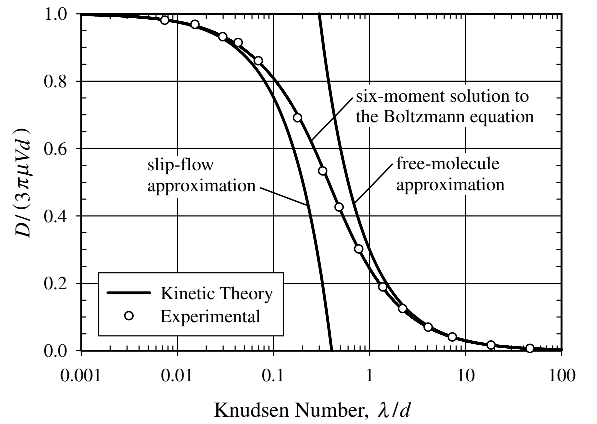

Fig. 1 Knudsen-number effect for the drag on a sphere in creeping flow.

The creeping-flow solution to the Boltzmann equation is independent of Mach number and asymptotically approaches the continuum solution in the limit as the Knudsen number approaches zero. If we choose a 1% deviation from the continuum solution as an approximate limiting boundary for continuum theory, then the analytical results presented in Fig. 1 suggest that for very low Reynolds numbers, continuum theory applies only for Knudsen numbers less than about 4×10^{-3}. In view of Eq. (4), this gives the approximate limiting relation

$$Kn = \sqrt{\pi\gamma/2}M/Re < \sim 4 \times 10^{-3} \tag{6}$$

After rearranging, the *creeping-flow limit for continuum theory* could be approximated from the relation

$$M < \sim 4 \times 10^{-3}\sqrt{2/(\pi\gamma)}Re, \quad \text{for } Re \to 0 \tag{7}$$

Although Eq. (7) strictly applies only in the limit as the Reynolds number approaches zero, comparison with experimental data has shown that creeping-flow analysis is generally valid for Reynolds numbers less than about 1.

III. Boundary-Layer Limit

Although the relation given by Eq. (7) agrees well with experimental data for very-low-Reynolds-number applications, such as flow about aerosol particles, it cannot be used for the high Reynolds numbers typically encountered in atmospheric flight. For objects moving through the atmosphere at high Reynolds numbers, continuum theory first begins to break down with failure of the no-slip boundary condition in the boundary layer near the leading edge of objects traveling at high altitudes and hypersonic airspeeds [20]. Classical continuum theory is based on the assumption that at an interface between a fluid and a solid wall, the fluid assumes the same velocity as the wall. However, the kinetic theory of gases predicts that for finite Knudsen numbers, the mean molecular velocity is not equal to the wall velocity at a fluid–wall interface.

A well-known solution from kinetic theory predicts the slip velocity at a gas–wall interface from what is commonly called the Chapman–Enskog expansion [9–14]. From this solution, which applies only for small Knudsen numbers, the tangential component of mean molecular velocity at a gas–wall interface can be expressed as

$$u|_{y=0} = u_w + \left(\frac{2}{\sigma} - 1\right)\lambda \left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{8}$$

where u is the component of mean molecular velocity tangent to the wall, y is the normal coordinate measured outward from the wall, u_w is the tangential component of the wall velocity, and σ is what is commonly called the accommodation coefficient [9–14]. Near the leading edge of a thin surface moving through the atmosphere at high Reynolds number, the normal component of the velocity gradient at the wall can be obtained from the well-known solution of Blasius [21]:

$$\frac{\nu}{V^2}\frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{0.332}{\sqrt{Re}} \tag{9}$$

After applying Eqs. (3) and (4), this is readily rearranged to give

$$\frac{\lambda}{V}\frac{\partial u}{\partial y}\bigg|_{y=0} \cong 0.4\sqrt{MKn} \tag{10}$$

Using Eq. (10) in Eq. (8) yields

$$\frac{u|_{y=0} - u_w}{V} \cong 0.4\left(\frac{2}{\sigma} - 1\right)\sqrt{MKn} \tag{11}$$

The relation given by Eq. (11) reduces to the classical no-slip continuum boundary condition in the limit as the Knudsen number approaches zero. If we once again choose a 1% deviation from the continuum result as our approximate limiting boundary for continuum theory, we have

$$0.4\left(\frac{2}{\sigma} - 1\right)\sqrt{MKn} < \sim 0.01 \tag{12}$$

Experiments with gases over various solid and liquid surfaces show that σ is slightly less than 1.0, usually between about 0.9 and 1.0 [18,19]. Thus, choosing a midrange value for σ and applying Eq. (4), the continuum constraint given by Eq. (12) can be written as

$$Kn = \sqrt{\pi\gamma/2}M/Re < \sim 5 \times 10^{-4}/M \tag{13}$$

After rearranging, the *boundary-layer limit for continuum theory* could be approximated from the relation

$$M < \sim\sqrt{5 \times 10^{-4}\sqrt{2/(\pi\gamma)}Re}, \quad \text{for } Re \to \infty \tag{14}$$

where the Mach number and Reynolds number are based on the freestream airspeed and the minimum significant characteristic length is measured parallel with the boundary layer. Although Eq. (14) strictly applies only in the limit as the Reynolds number approaches infinity, comparison with experimental data has shown that boundary-layer analysis is generally valid for Reynolds numbers greater than about 2000.

IV. Simple Approximation for the Transition

In the Reynolds-number transition region where neither the creeping-flow approximation nor the boundary-layer approximation is valid, it is very difficult to obtain analytical solutions to the Navier–Stokes equations. However, a very simple relation for interpolating between Eqs. (7) and (14) is given by

$$M < \sim\frac{4 \times 10^{-3}\sqrt{2/(\pi\gamma)}Re}{1 + 2\sqrt{8 \times 10^{-3}\sqrt{2/(\pi\gamma)}Re}} \tag{15}$$

Equation (15) has no analytical basis except for the fact that it exhibits the correct asymptotic behavior for small and large Reynolds numbers, as specified by Eqs. (7) and (14), respectively. This relation is shown graphically in Fig. 2 in comparison with its two asymptotes. Note that Eq. (15) agrees closely with Eq. (7) at Reynolds numbers below 1, and it agrees closely with Eq. (14) at Reynolds numbers above 2000. Also notice that this relation predicts that continuum theory can be applied to subsonic flows at any Reynolds number above 4000. At Reynolds numbers above 10^6, continuum theory is predicted to be valid for hypersonic Mach numbers up to 18 and above.

Results obtained from Eq. (15) predict that continuum theory can be applied for Mach-number and Reynolds-number combinations in the region below and to the right of the solid line in Fig. 2. For a given

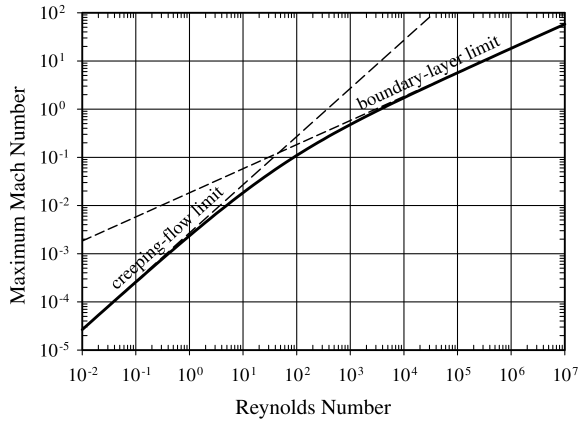

Fig. 2 Approximate Mach-number limits for continuum theory, as a function of Reynolds number.

altitude and Mach number, Eq. (15) could be used to determine a minimum permissible characteristic length. Similarly, for a given characteristic length and Mach number, Eq. (15) could be used to determine a maximum permissible altitude. For example, Eq. (15) predicts that at a Mach number of 0.1, continuum theory begins to break down for Reynolds numbers below about 90. Thus, for a Mach number of 0.1 at standard sea level, Eq. (15) predicts a minimum permissible characteristic length of 0.0015 in. (0.038 mm), which is less than one-half the thickness of a typical sheet of paper. At 100,000 ft, a Reynolds number of 90 with a Mach number of 0.1 corresponds to a minimum permissible characteristic length of 0.10 in. (2.5 mm). Similarly, for a Mach number of 0.1 and a characteristic length of 1 ft, the predicted maximum permissible altitude is about 217,000 ft.

The approximate continuum-theory constraint given by Eq. (15) can also be expressed as a relation between the Knudsen number and Mach number. Using Eq. (4) in Eq. (15) and rearranging yields

$$Kn < \sim 8 \times 10^{-3}[\sqrt{1 + 1/(2M)} - 1]^2 M \tag{16}$$

This relation is shown graphically in Fig. 3 compared with the creeping-flow and boundary-layer asymptotes, which are given by Eqs. (6) and (13), respectively.

Because the molecular mean free path in the standard atmosphere depends only on altitude, Eq. (16) can be used to predict the minimum significant characteristic dimension for the application of continuum theory as a function of Mach number and altitude. Such results are shown in Fig. 4. For low Mach numbers at sea level, the predicted minimum permissible significant characteristic dimension is about 16 μm. Even at 100,000 ft and Mach 10, the minimum permissible significant characteristic dimension is still less than 0.3 ft. Of course, these relations are only intended as a first approximation, so continuum theory should be used with caution near these limits.

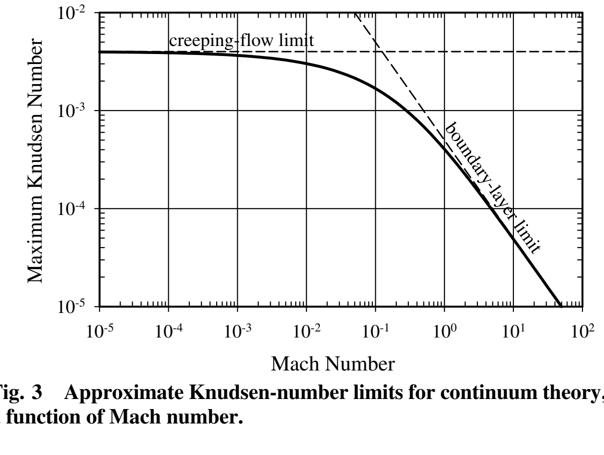

Fig. 3 Approximate Knudsen-number limits for continuum theory, as a function of Mach number.

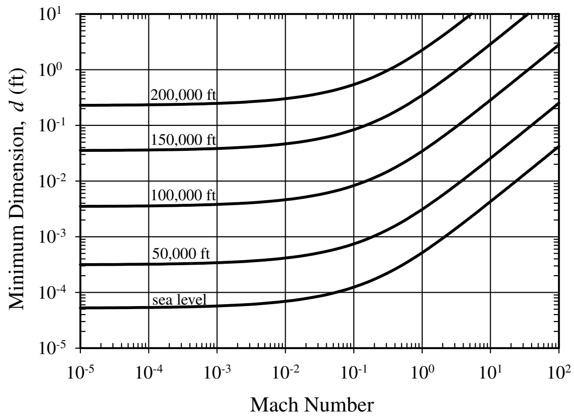

Fig. 4 Minimum significant dimension for continuum theory, as a function of Mach number and altitude.

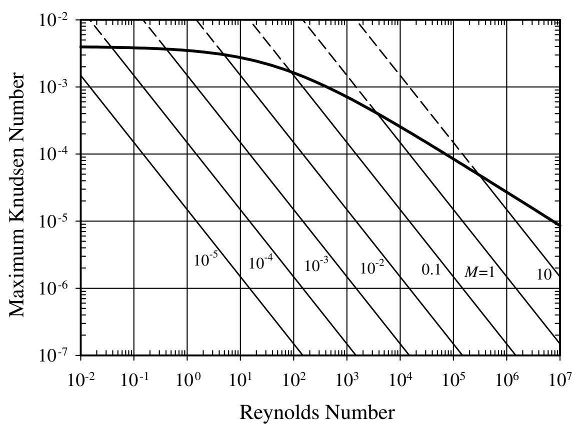

Fig. 5 Approximate Knudsen-number limits for continuum theory, as a function of Reynolds number.

The relation given by Eq. (4) can also be used to express the continuum constraint given by Eq. (15) as a relation between the Knudsen number and Reynolds number:

$$Kn < \sim \frac{4 \times 10^{-3}}{1 + 2\sqrt{8 \times 10^{-3}\sqrt{2/(\pi\gamma)}Re}} \tag{17}$$

This relation is shown graphically in Fig. 5 compared with several lines of constant Mach number. These Mach number contours are shown as solid lines in the region where continuum theory is predicted to be valid, and they are shown as dashed lines in the region where kinetic theory should be considered. For a given geometry, the Knudsen number increases with altitude. Thus, the vertical axis in Fig. 5 could be viewed as a dimensionless altitude. It is important to recognize that for fixed geometry and Mach number, the Reynolds number always decreases with increasing altitude.

V. Conclusions

The low-Reynolds-number limit for the approximate continuum-theory constraint expressed in Eqs. (15–17) is based on considerable analytical and experimental work that has been carried out in the field of aerosol technology. Thus, these constraints should provide reasonable results for estimating the limits of continuum theory at low Reynolds numbers.

At high Reynolds numbers, the approximate continuum constraint expressed in Eqs. (15–17) is based on the Blasius solution given by Eq. (9). This solution assumes a laminar boundary layer. Experiments have shown that boundary layers remain laminar only for Reynolds numbers up to about 10^6. At Reynolds numbers greater than one million, boundary layers typically transition to turbulent flow. Thus, we should not expect the continuum constraint developed here to apply at Reynolds numbers greater than 10^6. However, because the Reynolds number decreases with altitude for a fixed Mach number, Fig. 5 shows that the Reynolds number will be reduced to less than 10^6 before reaching the limiting altitude for continuum theory, except for high hypersonic Mach numbers. From Eq. (15), the limiting Mach number corresponding to a Reynolds number of 10^6 is about 18.2. Thus, Eqs. (15–17) should provide reasonable results for high Reynolds numbers and Mach numbers less than 18.

In the transition region between low and high Reynolds numbers, the approximate continuum-theory constraint presented here is based on a simple interpolation. Thus, Eqs. (15–17) should be used with greater caution for transition Reynolds numbers between about 1 and 2000. These relations are intended only as a first approximation, and continuum theory should always be used with caution near these limits.

References

[1] Anderson, J. D., "Continuum Versus Free Molecule Flow," *Fundamentals of Aerodynamics*, 4th ed., McGraw–Hill, New York, 2007, p. 58.

[2] Bertin, J. J., and Cummings, R. M., "Fluid as a Continuum," *Aerodynamics for Engineers*, 5th ed., Prentice–Hall, Upper Saddle River, NJ, 2009, pp. 7–8.

[3] Kuethe, A. M., and Chow, C. Y., "The Fluid Medium," *Foundations of Aerodynamics*, 5th ed., Wiley, New York, 1998, pp. 1–14.

[4] McCormick, B. W., "Fluid Mechanics," *Aerodynamics, Aeronautics, and Flight Mechanics*, 2nd ed., Wiley, New York, 1995, pp. 22–54.

[5] Fox, R. W., and McDonald, A. T., "Fluid as a Continuum," *Introduction to Fluid Mechanics*, 4th ed., Wiley, New York, 1992, pp. 17–18.

[6] Maxwell, J. C., "Illustrations of the Dynamical Theory of Gases—Part 1," *Philosophical Magazine*, Vol. 19, June 1860, pp. 19–32.

[7] Maxwell, J. C., "Illustrations of the Dynamical Theory of Gases—Part 2," *Philosophical Magazine*, Vol. 20, Dec. 1860, pp. 21–37.

[8] Maxwell, J. C., "On the Dynamical Theory of Gases," *Philosophical Transactions of the Royal Society of London*, Vol. 157, Jan. 1867, pp. 49–88.
doi:10.1098/rstl.1867.0004

[9] Jeans, J., *An Introduction to the Kinetic Theory of Gases*, Cambridge Univ. Press, Cambridge, England, U.K., 1940.

[10] Brush, S. G., *Kinetic Theory*, Vols. 1–3, Pergamon, Oxford, 1966.

[11] Liboff, R. L., *Kinetic Theory: Classical, Quantum, and Relativistic Descriptions*, Wiley, New York, 1998.

[12] Present, R. D., *Kinetic Theory of Gases*, McGraw–Hill, New York, 1958.

[13] Struchtrup, H., *Macroscopic Transport Equations for Rarefied Gas Flows*, Springer, New York, 2005.

[14] Vincenti, W. G., and Kruger, C. H., *Introduction to Physical Gas Dynamics*, Wiley, New York, 1965.

[15] Stokes, G. G., "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums," *Transactions of the Cambridge Philosophical Society*, Vol. 9, Jan. 1851, pp. 8–33.

[16] Phillips, W. F., "Drag on a Small Sphere Moving Through a Gas," *Physics of Fluids*, Vol. 18, No. 9, 1975, pp. 1089–1093.
doi:10.1063/1.861292

[17] Millikan, R. A., "The Isolation of an Ion, a Precision Measurement of its Charge, and the Correction of Stokes's Law," *Physical Review*, Series I, Vol. 32, No. 4, 1911, pp. 349–397.

[18] Millikan, R. A., "Coefficients of Slip in Gases and the Law of Reflection of Molecules from the Surfaces of Solids and Liquids," *Physical Review*, Series II, Vol. 21, No. 3, 1923, pp. 217–238.
doi:10.1103/PhysRev.21.217

[19] Millikan, R. A., "The General Law of Fall of a Small Spherical Body Through a Gas, and its Bearing upon the Nature of Molecular Reflection from Surfaces," *Physical Review*, Series II, Vol. 22, No. 1, 1923, pp. 1–23.
doi:10.1103/PhysRev.22.1

[20] White, F. M., "Kinetic Theory for Slip Velocity," and "Hypersonic Leading-Edge Shock-Wave-Boundary-Layer Interaction," *Viscous Fluid Flow*, 2nd ed., McGraw–Hill, New York, 1991, pp. 47–48, 535–539.

[21] Blasius, H., "Grenzschichten in Flüssikeiten mit Kleiner Reibung," *Zeitschrift für Angewandte Mathematik und Physik*, Vol. 56, Jan. 1908, pp. 1–37.